Single–peaks for a magnetic Schrödinger equation with critical growth

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Abstract

We prove existence results of complex-valued solutions for a semilinear Schrödinger equation with critical growth under the perturbation of an external electromagnetic field. Solutions are found via an abstract perturbation result in critical point theory, developed in [1, 2, 5].

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1 Introduction

This paper deals with some classes of elliptic equations which are perturbation of the time-dependent nonlinear Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -\hbar^2 \Delta \psi - |\psi|^{p-1}\psi \tag{1}$$

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under the effect of a magnetic field B_ε and an electric field E_ε whose sources are small in L^∞ sense. Precisely we will study the existence of wave functions $\psi\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ satisfying the nonlinear Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \left(\frac{\hbar}{i}\nabla - A_\varepsilon(x)\right)^2 \psi + W_\varepsilon(x)\psi - |\psi|^{p-1}\psi \tag{2}$$

where $A_\varepsilon(x)$ and $W_\varepsilon(x)$ are respectively a magnetic potential and an electric one, depending on a positive small parameter $\varepsilon > 0$. In the work, we assume that $A_\varepsilon(x) = \varepsilon\ A(x)$, $W_\varepsilon(x) = V_0 + \varepsilon^\alpha V(x)$, being $A\colon \mathbb{R}^N \to \mathbb{R}^N$ and $V_0 \in \mathbb{R}$, $V\colon \mathbb{R}^N \to \mathbb{R}$, $\alpha \in [1, 2]$.

On the right hand side of (2) the operator $\left(\frac{\hbar}{i}\nabla - A_\varepsilon\right)^2$ denotes the formal scalar product of the operator $\frac{\hbar}{i}\nabla - A_\varepsilon$ by itself, *i.e.*

$$\left(\frac{\hbar}{i}\nabla - A_\varepsilon(x)\right)^2 \psi := -\hbar^2 \Delta\psi - \frac{2\hbar}{i} A_\varepsilon \cdot \nabla\psi + |A_\varepsilon|^2 \psi - \frac{\hbar}{i}\psi \operatorname{div} A_\varepsilon$$

being $i^2 = -1$, $\hbar$ the Planck constant.

This model arises in several branches of physics, e.g. in the description of the Bose–Einstein condensates and in nonlinear optics (see [7, 11, 23, 25]).

If A is seen as the 1–form

$$A = \sum_{j=1}^N A_j dx_i,$$

then

$$B_\varepsilon = \varepsilon\, dA = \varepsilon \sum_{j<k} B_{jk}\, dx_j \wedge dx_k, \quad \text{where } B_{jk} = \partial_j A_k - \partial_k A_j,$$

represents the external magnetic field having source in εA (cf. [30]), while $E_\varepsilon = \varepsilon^\alpha \nabla V(x)$ is the electric field. The fixed $\hbar > 0$ the spectral theory of the operator has been studied in detail, particularly by Avron, Herbt, Simon [7] and Helffer [21, 22].

The search of standing waves of the type $\psi_\varepsilon(t, x) = e^{-iV_0\hbar^{-1}t} u_\varepsilon(x)$ leads to find a complex-valued solution $u\colon \mathbb{R}^N \to \mathbb{C}$ of the semilinear Schrödinger equation

$$\left(\frac{\hbar}{i}\nabla - \varepsilon A(x)\right)^2 u + \varepsilon^\alpha V(x) u = |u|^{p-1} u \quad \text{in } \mathbb{R}^N. \tag{3}$$

From a mathematical viewpoint, this equation has been studied in several papers in the *subcritical case* $1 < p < (N+2)/(N-2)$. In the pioneering paper [20], M. Esteban and P.L. Lions proved the existence of standing wave solutions to (2) in the case $V = 1$ identically, $\varepsilon > 0$ fixed, by a constrained minimization. Recently variational techniques are been employed to study equation (3) in the semiclassical limit ($\hbar \to 0^+$). We refer to [15, 17, 24, 27]. Recent results on multi-bumps solutions are obtained in [12] for bounded vector potentials and in [19] without any L^∞–restriction on $|A|$.

In the *critical case* $p = (N+2)/(N-2)$, we mention the paper [6] by Arioli and Szulkin where the potentials A and V are assumed to be periodic, $\varepsilon > 0$ fixed. The existence of a solution is proved whenever $0 \notin \sigma\left(\left(\frac{\nabla}{i} - A\right)^2 + V\right)$. We also cite the recent paper [13] by Chabrowski and Skulzin, dealing with entire solutions of (3).

In the present paper we are concerned with the critical case $p = (N+2)/(N-2)$, but V and A are not in general periodic potentials.

When the problem is nonmagnetic and static, i.e. $A = 0$, $V = 0$, and $\hbar = 1$ then problem (3) reduces to the equation

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad u \in D^{1,2}(\mathbb{R}^N, \mathbb{C}). \tag{4}$$

In Section 2 we prove that the least energy solutions to (4) are given by the functions $z = e^{i\sigma} z_{\mu,\xi}(x)$, where

$$z_{\mu,\xi}(x) = \kappa_N \frac{\mu^{(\frac{N}{2}-1)}}{(\mu^2 + |x-\xi|^2)^{\frac{N-2}{2}}}, \quad \kappa_N = (N(N-2))^{\frac{N-2}{4}} \tag{5}$$

and they correspond to the extremals of the Sobolev imbedding $D^{1,2}(\mathbb{R}^N, \mathbb{C}) \subset L^{2^*}(\mathbb{R}^N, \mathbb{C})$ (cf. Lemma 2.1).

The perturbation of (4) due to the action of an external magnetic potential A leads us to seek for complex–valued solutions. In general, the lack of compactness due to the critical growth of the nonlinear term produces several difficulties in facing the problem by global variational methods. We will attack (3) by means of a perturbation method in Critical Point Theory, see [1, 4, 5], and we prove the existence of a solution u_ε to (3) that is close for ε small enough to a solution of (4). After an appropriate finite dimensional reduction, we find that stable critical points on $]0, +\infty[\times\mathbb{R}^{N+1}$ of a suitable functional Γ correspond to points on $Z = \left\{ e^{i\sigma} z_{\mu,\xi} : \sigma \in S^1, \mu > 0, \xi \in \mathbb{R}^N \right\}$ from which there bifurcate solutions to (3) for $\varepsilon \neq 0$. If V changes its sign, we find at least two solutions to (3). The main result of the paper is Theorem 5.2, stated in Section 5.

We quote the papers [3, 14, 16], dealing with perturbed semilinear equations with critical growth without magnetic potential A.

Remark 1.1. It is apparent that the compact group S^1 acts on the space of solutions to (3). For simplicity, we will talk about solutions, rather than orbits of solutions.

Notation. The complex conjugate of any number $z \in \mathbb{C}$ will be denoted by $\bar{z}$. The real part of a number $z \in \mathbb{C}$ will be denoted by $\operatorname{Re} z$. The ordinary inner product between two vectors $a, b \in \mathbb{R}^N$ will be denoted by $a \cdot b$. We use the Landau symbols. For example $O(\varepsilon)$ is a generic function such that $\limsup_{\varepsilon\to 0} \frac{O(\varepsilon)}{\varepsilon} < \infty$, and $o(\varepsilon)$ is a function such that $\lim\limits_{\varepsilon\to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$. We will denote $D^{1,2}(\mathbb{R}^N, \mathbb{C}) = \left\{ u \in L^{2^*}(\mathbb{R}^N, \mathbb{C}) \mid \int_{\mathbb{R}^N} |\nabla u|^2 \, dx < \infty \right\}$, with a similar definition for $D^{1,2}(\mathbb{R}^N, \mathbb{R})$.

2 The limiting problem

Before proceeding, we recall some known facts about a couple of auxiliary problems. Recall that $2^* = 2N/(N-2)$.

(•) The problem

$$\begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N, \mathbb{R}). \end{cases} \tag{6}$$

possesses a smooth manifold of least-energy solutions

$$\widetilde{Z} = \left\{ z_{\mu,\xi} = \mu^{-\frac{(N-2)}{2}} z_0(\tfrac{x-\xi}{\mu}) \mid \mu > 0,\, \xi \in \mathbb{R}^N \right\} \tag{7}$$

where

$$z_0(x) = \kappa_N \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}}, \quad \kappa_N = (N(N-2))^{\frac{N-2}{4}}. \tag{8}$$

Explicitly,

$$z_{\mu,\xi}(x) = \kappa_N \mu^{-\frac{(N-2)}{2}} \frac{1}{\left(1+\left|\frac{x-\xi}{\mu}\right|^2\right)^{\frac{N-2}{2}}} = \kappa_N \frac{\mu^{(\frac{N}{2}-1)}}{(\mu^2+|x-\xi|^2)^{\frac{N-2}{2}}}. \tag{9}$$

These solutions are critical points of the Euler functional

$$\widetilde{f}_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} |u_+|^{2^*} dx, \tag{10}$$

defined on $D^{1,2}(\mathbb{R}^N,\mathbb{R}) \subset E$, and the following *nondegeneracy* property holds:

$$\ker \widetilde{f}_0''(z_{\mu,\xi}) = T_{z_{\mu,\xi}}\widetilde{Z} \quad \text{for all } \mu > 0,\, \xi \in \mathbb{R}^N. \tag{11}$$

(••) Similarly, $f_0 \in C^2(D^{1,2}(\mathbb{R}^N,\mathbb{C}))$ possesses a finite–dimensional manifold Z of least-energy critical points, given by

$$Z = \left\{ e^{i\sigma} z_{\mu,\xi} : \sigma \in S^1, \mu > 0, \xi \in \mathbb{R}^N \right\} \cong S^1 \times (0,+\infty) \times \mathbb{R}^N. \tag{12}$$

More precisely, following the ideas of [24] and [27], we give the following characterization.

Lemma 2.1. *Any least-energy solution to the problem*

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N,\mathbb{C}) \end{cases} \tag{13}$$

is of the form $u = e^{i\sigma} z_{\mu,\xi}$ for some suitable $\sigma \in [0,2\pi]$, $\mu > 0$ and $\xi \in \mathbb{R}^N$.

Proof. It is convenient to divide the proof into two steps.
Step 1: Let $z_0 = U$ the least energy solution associated to the energy functional (10) on the manifold

$$M_{0,r} = \left\{ v \in D^{1,2}(\mathbb{R}^N,\mathbb{R}) \setminus \{0\} \mid \int_{\mathbb{R}^N} |\nabla v|^2 \, dx = \int_{\mathbb{R}^N} |v|^{2^*} dx \right\}.$$

It is well-known that $z_0 = U$ is radially symmetric and unique (up to translation and dilation) positive solution to the equation (6). Let $b_{0,r} = b_r = \widetilde{f}_0(U) = \widetilde{f}_0(z_0)$. In a similar way, we define the class

$$M_{0,c} = \left\{ v \in E \setminus \{0\} \mid \int_{\mathbb{R}^N} |\nabla v|^2 \, dx = \int_{\mathbb{R}^N} |v|^{2^*} dx \right\}$$

and denote by $b_{0,c} = b_c = f_0(v)$ on $M_{0,c}$. Let $\sigma \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\mu > 0$,$\widetilde{v}(x) = z_{\mu,\xi}(x)$ positive solution to (6) and $\widetilde{U} = e^{i\sigma}\widetilde{v} = e^{i\sigma} z_{\mu,\xi}$ (i.e. $z_{\mu,\xi} = |\widetilde{U}(x)|$). It results that $\widetilde{U} = e^{i\sigma} z_{\mu,\xi}$ is a non-trivial least energy solution for $b_{0,c} = f_0(v)$ with $v \in M_{0,c}$.

Step 2: The following facts hold:

(i) $b_{0,c} = b_{0,r}$;

(ii) If $U_c = \widetilde{U}$ is a least energy solution of problem (13), then

$$|\nabla |U_c|(x)| = |\nabla U_c(x)| \quad \text{ and } \operatorname{Re}\left(i\overline{U_c}(x)\nabla U_c(x)\right) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N.$$

(iii) There exist $\sigma \in \mathbb{R}$ and a least energy solution $u_r : \mathbb{R}^N \to \mathbb{R}$ of problem (6) with

$$U_c(x) = e^{i\sigma} u_r(x) \quad \text{ for a.e. } x \in \mathbb{R}^N$$

or, equivalently, the least energy solution U_c for $b_{0,c}$ is the following

$$U_c(x) = e^{i\sigma} u_r(x) = e^{i\sigma} z_{\mu,\xi}(x) \quad \text{ for a.e. } x \in \mathbb{R}^N.$$

Observe that

$$b_{0,r} = \min_{v \in M_{0,r}} \widetilde{f}_0(v) \quad \text{and} \quad b_{0,c} = \min_{v \in M_{0,c}} f_0(v)$$

where $M_{0,r}$ and $M_{0,c}$ are the real and complex Nehari manifolds for $\widetilde{f}_0$ and f_0,

$$\begin{aligned} M_{0,r} &= \left\{ v \in D^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \mid \widetilde{f}_0'(v)[v] = 0 \right\} \\ &= \left\{ v \in D^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \mid \int_{\mathbb{R}^N} |\nabla v|^2 \, dx = \int_{\mathbb{R}^N} |v|^{2^*} \, dx \right\} \end{aligned}$$

and

$$\begin{aligned} M_{0,c} &= \{ v \in E \setminus \{0\} \mid f_0'(v)[v] = 0 \} \\ &= \left\{ v \in E \setminus \{0\} \mid \int_{\mathbb{R}^N} |\nabla v|^2 \, dx = \int_{\mathbb{R}^N} |v|^{2^*} \, dx \right\} \end{aligned}$$

So (i) is equivalent to

$$\begin{aligned} b_{0,r} &= \min_{v \in M_{0,r}} \widetilde{f}_0(v) = \widetilde{f}_0(u_r) \\ b_{0,c} &= \min_{v \in M_{0,c}} f_0(v) = f_0(U_c) \end{aligned}$$

Proof of (i)–(iii). Let $u \in E$ be given. For the sake of convenience, we introduce the functionals

$$\begin{aligned} T(u) &= \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \\ P(u) &= \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \, dx \end{aligned}$$

(resp. $\widetilde{T}(u)$ and $\widetilde{P}(u)$ as $u \in D^{1,2}(\mathbb{R}^N,\mathbb{R})$) such that $f_0(u) = \frac{1}{2}T(u) - P(u)$ as $u \in E$ (resp. $\widetilde{f}_0(u) = \frac{1}{2}\widetilde{T}(u) - \widetilde{P}(u)$ as $u \in D^{1,2}(\mathbb{R}^N,\mathbb{R})$).
Consider the following minimization problems

$$\begin{aligned}
\sigma_r &= \min\left\{\widetilde{T}(u) \mid u \in D^{1,2}(\mathbb{R}^N,\mathbb{R}), \widetilde{P}(u) = 1\right\} \\
\sigma_c &= \min\left\{T(u) \mid u \in E, P(u) = 1\right\}
\end{aligned}$$

Note that, obviously, there holds $\sigma_c \leq \sigma_r$. If we denote by u_* the Schwarz symmetric rearrangement (see [8]) of the positive real valued function $|u| \in D^{1,2}(\mathbb{R}^N,\mathbb{R})$, then, Cavalieri's principle yields

$$\int_{\mathbb{R}^N} |u_*|^{2^*}\, dx = \int_{\mathbb{R}^N} |u|^{2^*}\, dx$$

which entails $\widetilde{P}(u_*) = P(|u|)$. Moreover, by the Polya-Szëgö inequality, we have

$$\widetilde{T}(u_*) = \int_{\mathbb{R}^N} |\nabla u_*|^2\, dx \leq \int_{\mathbb{R}^N} |\nabla |u||^2\, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\, dx = T(u)$$

where the second inequality follows from the following diamagnetic inequality

$$\int_{\mathbb{R}^N} |\nabla |u||^2\, dx \leq \int_{\mathbb{R}^N} |D^\varepsilon |u||^2\, dx \quad \text{for all } u \in H^\varepsilon_{A,V}$$

with $D^\varepsilon = \frac{\nabla}{i} - \varepsilon A$ and $A = 0$. Therefore, one can compute σ_c by minimizing over the subclass of positive, radially symmetric and radially decreasing functions $u \in D^{1,2}(\mathbb{R}^N,\mathbb{R})$. As a consequence, we have $\sigma_r \leq \sigma_c$. In conclusion, $\sigma_r = \sigma_c$. Observe now that

$$\begin{aligned}
b_{0,r} &= \min\left\{\widetilde{f}_0(u) \mid u \in D^{1,2}(\mathbb{R}^N,\mathbb{R}) \setminus \{0\} \text{ is a solution to (6)}\right\}, \\
b_{0,c} &= \min\left\{f_0(u) \mid u \in E \setminus \{0\} \text{ is a solution to (13)}\right\}.
\end{aligned}$$

The above inequalities hold since any nontrivial real (resp. complex) solution of (6) (resp. (13)) belongs to $M_{0,r}$ (resp. $M_{0,c}$) and, conversely, any solution of $b_{0,r}$ (resp. $b_{0,c}$) produces a nontrivial solution of (6) (resp. (13)). Moreover, it follows from an easy adaptation of [8, Th. 3] that $b_{0,r} = \sigma_r$ as well as $b_{0,c} = \sigma_c$. In conclusion, there holds

$$b_{0,r} = \sigma_r = b_{0,c} = \sigma_c$$

which proves (i).

To prove (ii), let $U_c\colon \mathbb{R}^N \to \mathbb{C}$ be a least energy solution to problem (13) and assume by contradiction that

$$\mathcal{L}^N\left(\left\{x \in \mathbb{R}^N : |\nabla |U_c|| < |\nabla U_c|\right\}\right) > 0$$

where $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$. Then, we would get $\widetilde{P}(|U_c|) = P(U_c)$ and

$$\widetilde{P}(|U_c|) = \frac{1}{2^*}\int_{\mathbb{R}^N} |U_c|^{2^*}\, dx = \frac{1}{2^*}\int_{\mathbb{R}^N} |U_c|^{2^*}\, dx = P(U_c)$$

and

$$\sigma_r \leq \int_{\mathbb{R}^N} |\nabla |U_c||^2\, dx < \int_{\mathbb{R}^N} |\nabla U_c|^2\, dx = \sigma_c$$

which is a contradiction. The second assertion in (ii) follows by direct computations. Indeed, a.e. in $\mathbb{R}^N$, we have

$$|\nabla|U_c|| = |\nabla U_c| \quad \text{if and only if } \operatorname{Re} U_c \left(\nabla \operatorname{Im} U_c\right) = \operatorname{Im} U_c \nabla \left(\operatorname{Re} U_c\right).$$

If this last condition holds, in turn, a.e. in $\mathbb{R}^N$, we have

$$\overline{U_c}\nabla U_c = \operatorname{Re} U_c \nabla \left(\operatorname{Re} U_c\right) + \operatorname{Im} U_c \nabla \left(\operatorname{Im} U_c\right)$$

which implies the desired assertion.

Finally, the representation formula of (iii) $U_c(x) = e^{i\sigma} u_r(x)$ is an immediate consequence of (ii), since one obtains $U_c = e^{i\sigma}|U_c|$ for some $\sigma \in \mathbb{R}$. $\square$

Remark 2.2. For the reader's convenience, we write here the second derivative of f_0 at any $z \in Z$:

$$\langle f_0''(z)v, w\rangle_E = \operatorname{Re}\int_{\mathbb{R}^N} \nabla v \cdot \overline{\nabla w}\, dx - \operatorname{Re}\int_{\mathbb{R}^N} |z|^{2^*-2} v\overline{w}\, dx \\ - \operatorname{Re}(2^*-2)\int_{\mathbb{R}^N} |z|^{2^*-4} \operatorname{Re}(z\overline{v}) z\overline{w}\, dx. \quad (14)$$

In particular, $f_0''(z)$ can be identified with a compact perturbation of the identity operator.

We now come to the most delicate requirement of the perturbation method.

Lemma 2.3. *For each $z = e^{i\sigma} z_{\mu,\xi} \in Z$, there holds*

$$T_z Z = \ker f_0''(z) \qquad \text{for all } z \in Z,$$

where

$$T_{e^{i\sigma} z_{\mu,\xi}} Z = \operatorname{span}_{\mathbb{R}}\left\{\frac{\partial e^{i\sigma} z_{\mu,\xi}}{\partial \xi_1}, \ldots, \frac{\partial e^{i\sigma} z_{\mu,\xi}}{\partial \xi_N}, \frac{\partial e^{i\sigma} z_{\mu,\xi}}{\partial \mu}, \frac{\partial e^{i\sigma} z_{\mu,\xi}}{\partial \sigma} = ie^{i\sigma} z_{\mu,\xi}\right\}. \quad (15)$$

Proof. The inclusion $T_z Z \subset \ker f_0''(z)$ is always true, see [1]. Conversely, we prove that for any $\varphi \in \ker f_0''(z)$ there exist numbers $a_1, \ldots, a_N$, b, $d \in \mathbb{R}$ such that

$$\varphi = \sum_{j=1}^N a_j \frac{\partial e^{i\sigma} z_{\mu,\xi}}{\partial \xi_j} + b\frac{\partial e^{i\sigma} z_{\mu,\xi}}{\partial \mu} + die^{i\sigma} z_{\mu,\xi}. \quad (16)$$

If we can prove the following representation formulæ, then (16) will follow.

$$\operatorname{Re}(\varphi\overline{e^{i\sigma}}) = \sum_{j=1}^N a_j \frac{\partial z_{\mu,\xi}}{\partial \xi_j} + b\frac{\partial z_{\mu,\xi}}{\partial \mu} \quad (17)$$

$$\operatorname{Im}(\varphi\overline{e^{i\sigma}}) = d z_{\mu,\xi}. \quad (18)$$

We will use a well-known result for the scalar case:

$$\ker \widetilde{f}_0''(z_{\mu,\xi}) \equiv T_{z_{\mu,\xi}} \widetilde{Z} = \operatorname{span}_{\mathbb{R}}\left\{\frac{\partial z_{\mu,\xi}}{\partial \xi_1}, \ldots, \frac{\partial z_{\mu,\xi}}{\partial \xi_N}, \frac{\partial z_{\mu,\xi}}{\partial \mu}\right\}$$

Step 1: proof of (17). We wish to prove that $\mathrm{Re}(\varphi\overline{e^{i\sigma}}) \in \ker \widetilde{f}_0''(z_{\mu,\xi})$. Recall that $\varphi \in \ker f_0''(e^{i\sigma}z_{\mu,\xi})$, so

$$\langle f_0''(e^{i\sigma}z_{\mu,\xi})\varphi, \psi\rangle = 0 \quad \text{for all } \psi \in E. \tag{19}$$

Select $\psi = e^{i\sigma}v$, with $v \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$.

$$\begin{aligned} 0 = \langle f_0''(e^{i\sigma}z_{\mu,\xi})\varphi, ve^{i\sigma}\rangle &= \mathrm{Re}\int \nabla(\varphi e^{-i\sigma})\nabla v \\ &- (2^*-2)\int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2}\,\mathrm{Re}(\overline{e^{i\sigma}}\varphi)v - \int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2}\,\mathrm{Re}(\overline{e^{i\sigma}}\varphi)v \\ = \int_{\mathbb{R}^N} \nabla(\mathrm{Re}(\varphi\overline{e^{i\sigma}})\nabla v &- (2^*-1)\int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2}\,\mathrm{Re}(\overline{e^{i\sigma}}\varphi)v = \langle \widetilde{f}_0''(z_{\mu,\xi})\,\mathrm{Re}(\varphi\overline{e^{i\sigma}}), v\rangle. \end{aligned}$$

This implies that

$$\mathrm{Re}(\overline{e^{i\sigma}}\varphi) \in \ker \widetilde{f_0''}(z_{\mu,\xi}) \equiv T_{z_{\mu,\xi}}\widetilde{Z}$$

from which it follows

$$\mathrm{Re}(\varphi\overline{e^{i\sigma}}) = \sum_{j=1}^{N} a_j \frac{\partial z_{\mu,\xi}}{\partial \xi_j} + b\frac{\partial z_{\mu,\xi}}{\partial \mu}$$

for some real constants $a_1, \dots, a_N$ and b.

Step 2: proof of (18)**.** Test (19) on $\psi = ie^{i\sigma}w \in E$ with $w : \mathbb{R}^N \to \mathbb{R}$. We get

$$\begin{aligned} 0 &= \langle f_0''(e^{i\sigma}z_{\mu,\xi})\varphi, ie^{i\sigma}w\rangle = \mathrm{Re}\int_{\mathbb{R}^N} \nabla(-i\varphi e^{-i\sigma})\cdot\nabla w - \mathrm{Re}\int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2}(-i\varphi e^{-i\sigma})w \\ &\qquad [\text{being } \mathrm{Re}(-i\varphi e^{-i\sigma}) = \mathrm{Im}(\varphi e^{-i\sigma})] \\ &= \int_{\mathbb{R}^N} \nabla(\mathrm{Im}(\varphi e^{-i\sigma}))\cdot\nabla w - \int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2}\,\mathrm{Im}(\varphi e^{-i\sigma})w \\ &= \int_{\mathbb{R}^N} \nabla(\mathrm{Im}(\varphi\overline{e^{i\sigma}}))\cdot\nabla w - \int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2}\left[\mathrm{Im}(\varphi\overline{e^{i\sigma}})\right]_+ w. \end{aligned} \tag{20}$$

We can take $\mu = 1$ and $\xi = 0$, otherwise we perform the change of variable $x \mapsto \mu x + \xi$.

From (20) we get that $u := \mathrm{Im}(\varphi\overline{e^{i\sigma}})$ satisfies the equation

$$-\Delta u = \frac{N(N-2)}{(1+|x|^2)^2}u \quad \text{in } D^{-1,2}(\mathbb{R}^N, \mathbb{R}). \tag{21}$$

We will study this linear equation by an inverse stereographic projections onto the sphere S^N. Precisely, for each point $\xi \in S^N$, denote by x its corresponding point under the stereographic projection π from S^N to $\mathbb{R}^N$, sending the north pole on S^N to ∞. That is, suppose $\xi = (\xi_1, \xi_2, \dots, \xi_{N+1})$ is a point in S^N, $x = (x_1, \dots, x_N)$, then $\xi_i = \frac{2x_i}{1+|x|^2}$ for $1 \le i \le N$; $\xi_{N+1} = \frac{|x|^2-1}{|x|^2+1}$.

Recall that, on a Riemannian manifold (M, g), the conformal Laplacian is defined by

$$L_g = -\Delta_g + \frac{N-2}{4(N-1)}S_g,$$

where $-\Delta_g$ is the Laplace–Beltrami operator on M and S_g is the scalar curvature of (M, g). It is known that

$$L_g(\Phi(u)) = \varphi^{-\frac{N+2}{N-2}} L_\delta(u),$$

where δ is the euclidean metric of $\mathbb{R}^N$, $\varphi(x) = \left(\frac{2}{1+|x|^2}\right)^{(N-2)/2}$ and

$$\Phi\colon D^{1,2}(\mathbb{R}^N) \to H^1(S^n), \qquad \Phi(u)(x) = \frac{u(\pi(x))}{\varphi(\pi(x))}$$

is an isomorphism between $H^1(S^n)$ and $E := D^{1,2}(\mathbb{R}^N)$. Therefore, if $U = \Phi(u)$, then (21) changes into the equation

$$-\Delta_{g_0} U + \frac{N-2}{4(N-1)} S_{g_0} U = \frac{N(N-2)}{4} U, \tag{22}$$

where g_0 is the standard riemannian metric on S^N, and

$$S_{g_0} = N(N-1)$$

is the constant scalar curvature of (S^N, g_0). As a consequence, (22) implies that

$$-\Delta_{g_0} U = 0,$$

i.e. U is an eigenfunction of $-\Delta_{g_0}$ corresponding to the eigenvalue $\lambda = 0$. But the point spectrum of $-\Delta_{g_0}$ is completely known (see [9, 10]), consisting of the numbers

$$\lambda_k = k(k+N-1), \qquad k = 0, 1, 2, \dots$$

with associated eigenspaces of dimension

$$\frac{(N+k-2)!\,(N+2k-1)}{k!\,(N-1)!}.$$

Hence we deduce that $k = 0$, and U belongs to an eigenspace of dimension 1. Since $z_{\mu,\xi}$ is a solution to (21), we conclude that there exists $d \in \mathbb{R}$ such that

$$\mathrm{Im}(\varphi\overline{e^{i\sigma}}) = d z_{\mu,\xi}.$$

This completes the proof. $\square$

3 The functional framework

In the variational framework of the problem, solutions to (3) can be found as critical points of the energy functional $f_\varepsilon : E \to \mathbb{R}$ defined by

$$f_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left| \left(\frac{\nabla}{i} - \varepsilon A(x)\right) u \right|^2 dx + \frac{\varepsilon^\alpha}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx, \tag{23}$$

on the real Hilbert space

$$E = D^{1,2}(\mathbb{R}^N, \mathbb{C}) = \left\{ v \in L^{2^*}(\mathbb{R}^N, \mathbb{C}) \mid \int_{\mathbb{R}^N} |\nabla v|^2 dx < \infty \right\} \tag{24}$$

endowed with the inner product

$$\langle u, v \rangle_E = \mathrm{Re} \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v}\, dx. \tag{25}$$

We shall assume throughout the paper that

(N) $N > 4$,

(A1) $A \in C^1(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^N) \cap L^r(\mathbb{R}^N, \mathbb{R}^N)$ with $1 < r < N$

(A2) $\operatorname{div} A \in L^{N/2}(\mathbb{R}^N, \mathbb{R})$,

(V) $V \in C(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty(\mathbb{R}^N, \mathbb{R}) \cap L^s(\mathbb{R}^N, \mathbb{R})$, with $1 < s < N/2$.

The functional f_ε is well defined on E. Indeed,

$$\int_{\mathbb{R}^N} \left| \left(\frac{\nabla}{i} - \varepsilon A(x) \right) u \right|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \varepsilon^2 \int_{\mathbb{R}^N} |A|^2 |u|^2 - \mathrm{Re} \int_{\mathbb{R}^N} \frac{\nabla u}{i} \cdot \varepsilon A\, \overline{u},$$

and all the integrals are finite by virtue of **(A1)**. Moreover, $f_\varepsilon \in C^2(E, \mathbb{R})$.

In this section, we perform a finite–dimensional reduction on f_ε according to the methods of [1, 5]. Roughly speaking, since the unperturbed problem (i.e. (3) with $\varepsilon = 0$) has a whole C^2 manifold of critical points, we can deform this manifold is a suitable manner and get a *finite–dimensional natural constraint* for the Euler–Lagrange functional associated to (3). As a consequence, we can find solutions to (3) in correspondence to (stable) critical points of an auxiliary map — called the Melnikov function — in finite dimension.

Now we focus on the case $\alpha = 2$, as in the other cases $\alpha \in [1, 2[$ the magnetic potential A no longer affects the finite-dimensional reduction (see Remark (5.3)).

So that we can write the functional f_ε as

$$f_\varepsilon(u) = f_0(u) + \varepsilon G_1(u) + \varepsilon^2 G_2(u) \tag{26}$$

where

$$f_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*}, \tag{27}$$

$$G_1(u) = -\,\mathrm{Re}\, \frac{1}{i} \int_{\mathbb{R}^N} \nabla u \cdot A\, \overline{u}, \quad G_2(u) = \frac{1}{2} \int_{\mathbb{R}^N} |A|^2 |u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |u|^2. \tag{28}$$

We can now use the arguments of [1, 5] to build a natural constraint for the functional f_ε.

Theorem 3.1. *Given $R > 0$ and $B_R = \{u \in E : ||u|| \le R\}$, there exist ε_0 and a smooth function $w = w(z, \varepsilon) = w(e^{i\sigma} z_{\mu,\xi}, \varepsilon) = w(\sigma, \mu, \xi, \varepsilon)$, $w(z, \varepsilon) : M = Z \cap B_R \times (\varepsilon_0, \varepsilon_0) \to E$ such that*

1. $w(z,0)=0$ *for all* $z\in Z\cap B_R$

2. $w(z,\varepsilon)$ *is orthogonal to* T_zZ, *for all* $(z,\varepsilon)\in M$. *Equivalently* $w(z,\varepsilon)\in(T_zZ)^\perp$

3. *the manifold* $Z_\varepsilon=\{z+w(z,\varepsilon):(z,\varepsilon)\in M\}$ *is a natural constraint for* f'_ε: *if* $u\in Z_\varepsilon$ *and* $f'_\varepsilon|_{Z_\varepsilon}=0$, *then* $f'_\varepsilon(u)=0$.

For future reference let us recall that w satisfies 2. above and $Df_\varepsilon(z+w)\in T_zZ$, namely $f''_0(z)[w]+\varepsilon G'_1(z)+o(\varepsilon)\in T_zZ$. As a consequence, if $G'_1(z)\perp T_zZ$ (to be proved as Lemma 3.2), one finds

$$w(\varepsilon,z)=-\varepsilon L_zG'_1(z)+o(\varepsilon), \tag{29}$$

where L_z denotes the inverse of the restriction to $(T_zZ)^\perp$ of $f''_0(z)$.

Lemma 3.2. $G_1(z)=0$ *for all* $z\in Z$.

Proof.

$$\begin{aligned} G_1(z) &= -\operatorname{Re}\int_{\mathbb{R}^N}\frac{\nabla z}{i}\cdot A(x)\,\overline{z}\,dx=\left[z=e^{i\sigma}z_{\mu,\xi}\right]\\ &= -\operatorname{Re}\int_{\mathbb{R}^N}e^{i\sigma}\frac{\nabla z_{\mu,\xi}}{i}\cdot A(x)\,e^{-i\sigma}\overline{z_{\mu,\xi}}\,dx=\\ &= -\operatorname{Re}\int_{\mathbb{R}^N}\frac{\nabla z_{\mu,\xi}}{i}\cdot A(x)\,z_{\mu,\xi}\,dx=0. \end{aligned}$$

□

Hence we cannot hope to apply directly the tools contained in [1], since the *Melnikov function* would vanish identically. However, following [4], we can find a slightly implicit Melnikov function whose stable critical points produce critical points of f_ε.

Lemma 3.3. *Let* $\Gamma:Z\to\mathbb{R}$ *be defined by setting*

$$\Gamma(z)=G_2(z)-\frac{1}{2}\left(L_zG'_1(z),G'_1(z)\right). \tag{30}$$

Then we have

$$f_\varepsilon(z+w(\varepsilon,z))=f_0(z)+\varepsilon^2\Gamma(z)+o(\varepsilon^2). \tag{31}$$

Proof. Since $G_1|_Z\equiv 0$, then $G'_1(z)\in(T_zZ)^\perp$. Then one finds

$$\begin{aligned} f_\varepsilon(z+w(\varepsilon,z)) &= f_0(z+w(\varepsilon,z))+\varepsilon G_1(z+w(\varepsilon,z))+\varepsilon^2G_2(z+w(\varepsilon,z))\\ &= f_0(z)+\frac{1}{2}f''_0(z)[w,w]+\varepsilon G_1(z)+\varepsilon G'_1(z)[w]+\varepsilon^2G_2(z)+o(\varepsilon^2). \end{aligned}$$

Using (29) and Lemma 3.2 the lemma follows. □

Remark 3.4. We notice that $\Gamma=G_2(z)+\frac{1}{2}\left(G'_1(z),\phi\right)$, where z stands for $e^{i\sigma}z_{\mu,\xi}$ and $\phi=\lim_{\varepsilon\to0}\frac{w}{\varepsilon}$.

Remark 3.5. By the definition of $z \in Z$, it results: $\Gamma(z) = \Gamma(e^{i\sigma} z_{\mu,\xi}) = \Gamma(\sigma, \mu, \xi)$. In the sequel, we will write freely $\Gamma(\sigma, \mu, \xi) \equiv \Gamma(\mu, \xi)$ since Γ is σ-invariant. Indeed, it is easy to check that G_2 is σ-invariant. In fact, by the definition of $G_2(z)$ and $z = e^{i\sigma} z_{\mu,\xi}$, it results:

$$G_2(\sigma, \mu, \xi) = G_2(e^{i\sigma} z_{\mu,\xi}) = \frac{1}{2} \int |A(x)|^2 |z_{\mu,\xi}|^2 \, dx + \frac{1}{2} \int |V(x)||z_{\mu,\xi}|^2 \, dx \equiv G_2(\mu, \xi).$$

It remains to prove that $\langle G_1'(z), \phi \rangle$ is σ-invariant. We will show that $\phi = e^{i\sigma} \psi(\mu, \xi)$ with $\psi(\mu, \xi) \in \mathbb{C}$ independent on σ which immediately gives

$$\begin{aligned} \langle G_1'(e^{i\sigma} z_{\mu,\xi}), \phi \rangle &= -\operatorname{Re} \int \frac{1}{i} e^{i\sigma} \nabla z_{\mu,\xi} \cdot A(x) e^{-i\sigma} \overline{\psi(\mu, \xi)} \, dx \\ &\quad - \operatorname{Re} \int \frac{1}{i} \nabla \psi_{\mu,\xi} \cdot A(x) z_{\mu,\xi} \, dx = \langle G_1'(z_{\mu,\xi}), \psi(\mu, \xi) \rangle . \end{aligned}$$

We begin to recall that $\phi = \lim_{\varepsilon \to 0^+} \frac{w(\varepsilon, z)}{\varepsilon}$, where $w(\varepsilon, z)$ is such that

$$f_\varepsilon'(e^{i\sigma} z_{\mu,\xi} + w(\sigma, \mu, \xi)) \in T_{e^{i\sigma} z_{\mu,\xi}} Z.$$

By (15), this condition means that

$$f_\varepsilon'(e^{i\sigma} z_{\mu,\xi} + w(\sigma, \mu, \xi)) = \sum_{i=1}^{N} a_i e^{i\sigma} \frac{\partial z_{\mu,\xi}}{\partial \xi_i} + b e^{i\sigma} \frac{\partial z_{\mu,\xi}}{\partial \mu} + d e^{i\sigma} i z_{\mu,\xi}, \tag{32}$$

with $a_1, \ldots, a_N$, b, d, $\in \mathbb{R}$.

Let $w(\sigma, \mu, \xi) = e^{i\sigma} \widetilde{w}$ with $\widetilde{w} \in D^{1,2}(\mathbb{R}^N, \mathbb{C})$. Testing (32) by $e^{i\sigma} v(x)$ with $v \in D^{1,2}(\mathbb{R}^N, \mathbb{C})$, we derive that $z_{\mu,\xi} + \widetilde{w}$ is a solution of an equation independently on σ. Thus, also $\widetilde{w}$ is independent on σ and it can be denoted as $\widetilde{w}(\mu, \xi)$. Set $\psi(\mu, \xi) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{w}(\mu, \xi)}{\varepsilon}$, we deduce that $\phi = e^{i\sigma} \psi(\mu, \xi)$.

4 Asymptotic study of Γ

In order to find critical points of Γ it is convenient to study the behavior of Γ as $\mu \to 0$ and as $\mu + |\xi| \to \infty$. Our goal is to show:

Proposition 4.1. *Γ can be extended smoothly to the hyperplane $\{(0, \xi) \in \mathbb{R} \times \mathbb{R}^N\}$ by setting*

$$\Gamma(0, \xi) = 0. \tag{33}$$

Moreover there results

$$\Gamma(\mu, \xi) \to 0, \quad \text{as } \mu + |\xi| \to +\infty. \tag{34}$$

The proof of this Proposition is rather technical, so we split it into several lemmas in which we will use the formulation of $\Gamma = G_2(z) + \frac{1}{2} (G_1'(z), \phi)$, where $\phi = \lim_{\varepsilon \to 0} \frac{w}{\varepsilon}$.

Lemma 4.2. *Under assumption **(A1)** there holds*

$$\lim_{\mu \to 0^+} \frac{1}{2} \int_{\mathbb{R}^N} |A(x)|^2 |z_{\mu,\xi}|^2 dx = 0. \tag{35}$$

Proof. Let $z = e^{i\sigma} z_{\mu,\xi} \in Z$. Then

$$\begin{aligned} H_2(z) &= \frac{1}{2}\int_{\mathbb{R}^N} |A(x)|^2 |z_{\mu,\xi}|^2 dx \qquad (36) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |A(x)|^2 \left(\kappa_N \mu^{-\frac{(N-2)}{2}} \left(1 + \left|\frac{x-\xi}{\mu}\right|^2\right)^{\frac{2-N}{2}} \right)^2 dx \\ &= \frac{\kappa_N^2}{2\mu^{(N-2)}} \int_{\mathbb{R}^N} \frac{|A(x)|^2}{\left(1 + \left|\frac{x-\xi}{\mu}\right|^2\right)^{N-2}} dx \end{aligned}$$

Using the change of variable $y = \frac{x-\xi}{\mu}$, or $x = \mu y + \xi$, we can write

$$\begin{aligned} H_2(z) &= \frac{\kappa_N^2}{2\mu^{(N-2)}} \int_{\mathbb{R}^N} |A(\mu y + \xi)|^2 \frac{1}{(1+|y|^2)^{N-2}} \mu^N dy \\ &= \frac{\kappa_N^2}{2}\mu^2 \int_{\mathbb{R}^N} \frac{|A(\mu y + \xi)|^2}{(1+|y|^2)^{N-2}} dy \end{aligned}$$

and using the hypothesis **(A1)**

$$H_2(z) \le \mu^2 C_N \|A\|_\infty^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}} dy, \qquad (37)$$

the lemma follows. □

The proof of the following Lemma is similar and thus omitted.

Lemma 4.3. *Under assumption* ***(V)*** *there holds*

$$\lim_{\mu \to 0^+} \frac{1}{2} \int_{\mathbb{R}^N} V(x) |z_{\mu,\xi}|^2 dx = 0. \qquad (38)$$

Lemma 4.4. *There holds*

$$\lim_{\mu \to 0^+} \langle G_1'(z), \phi \rangle = 0. \qquad (39)$$

Proof. We write

$$\langle G_1'(z), \phi \rangle_E = \alpha_1 + \alpha_2,$$

where

$$\begin{aligned} \alpha_1 &= -\operatorname{Re} \int_{\mathbb{R}^N} \frac{\nabla z}{i} \cdot A(x) \overline{\phi}\, dx \qquad (40) \\ \alpha_2 &= -\operatorname{Re} \int_{\mathbb{R}^N} \frac{\nabla \phi}{i} \cdot A(x) \overline{z}\, dx. \qquad (41) \end{aligned}$$

It is convenient to introduce $\phi^*(y)$ by setting

$$\phi^*(y) = \phi^*_{\mu,\xi}(y) = \mu^{\frac{N}{2}-1} \phi(\mu y + \xi)$$

Using the expression of $z = e^{i\sigma}\mu^{-\frac{(N-2)}{2}}z_0(\frac{x-\xi}{\mu})$ and the change of variable $x = \mu y + \xi$ we can write:

$$\begin{aligned}
\alpha_1 &= -\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}\nabla_x e^{i\sigma}\mu^{-\frac{(N-2)}{2}} z_0\left(\frac{x-\xi}{\mu}\right)\cdot A(x)\overline{\phi(x)}\,dx \\
&= -\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}e^{i\sigma}\nabla_y z_0(y)\mu^{\frac{N}{2}}\cdot A(\mu y+\xi)\overline{\phi(\mu y+\xi)}\,dy \\
&= -\mu\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}e^{i\sigma}\nabla_y z_0(y)\cdot A(\mu y+\xi)\overline{\phi^*(y)}\,dy
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2 &= -\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}\nabla_x\phi(x)\cdot A(x)e^{-i\sigma}\mu^{-\frac{(N-2)}{2}} z_0\left(\frac{x-\xi}{\mu}\right)dx \\
&= -\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}\nabla_y\phi(\mu y+\xi)\mu^{-1}\cdot A(\mu y+\xi)e^{-i\sigma}\mu^{-\frac{(N-2)}{2}}\mu^N z_0(y)\,dy \\
&= -\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}\nabla\phi(\mu y+\xi)\mu^{-1}\cdot A(\mu y+\xi)e^{-i\sigma}\mu^{\frac{N}{2}+1}z_0(y)\,dy \\
&= -\mu\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}\nabla\phi^*(y)\cdot A(\mu y+\xi)e^{-i\sigma}z_0(y)\,dy.
\end{aligned}$$

Now the conclusion follows easily from the next lemma. □

Lemma 4.5. *As* $\mu \to 0^+$,

$$\phi^*_{\mu,\xi} \to 0 \quad \textit{strongly in } E. \tag{42}$$

Proof. For all $v \in E$, due to the divergence theorem, we have

$$\begin{aligned}
\langle G_1'(z), v\rangle_E &= -\operatorname{Re}\int_{\mathbb{R}^N}\frac{\nabla z}{i}\cdot A(x)\overline{v}\,dx - \operatorname{Re}\int_{\mathbb{R}^N}\frac{\nabla v}{i}\cdot A(x)\overline{z}\,dx \\
&= -\operatorname{Re}\int_{\mathbb{R}^N}\frac{\nabla z}{i}\cdot A(x)\overline{v}\,dx - \operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}\sum_{j=1}^N\frac{\partial v}{\partial x_j}A_j(x)\overline{z}\,dx \\
&= -\operatorname{Re}\int_{\mathbb{R}^N}\frac{\nabla z}{i}\cdot A(x)\overline{v}\,dx + \operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}\sum_{j=1}^N v\frac{\partial}{\partial x_j}(A_j\overline{z})\,dx \\
&= -\operatorname{Re}\int_{\mathbb{R}^N}\frac{\nabla z}{i}\cdot A(x)\overline{v}\,dx + \operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}v\operatorname{div}A\,\overline{z}\,dx + \operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}vA\cdot\overline{\nabla z}\,dx \\
&= -2\operatorname{Re}\int_{\mathbb{R}^N}\frac{\nabla z}{i}\cdot A(x)\overline{v}\,dx - \operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}\operatorname{div}A\,z\overline{v}\,dx
\end{aligned}$$

where the last integral is finite by assumption **(A2)** and

$$\begin{aligned}
(f_0''(z)w_{\mu,\xi}, v) = \operatorname{Re}\int_{\mathbb{R}^N}\nabla w_{\mu,\xi}\cdot\overline{\nabla v}\,dx - \operatorname{Re}\int_{\mathbb{R}^N}|z|^{2^*-2}w_{\mu,\xi}\overline{v}\,dx \\
- \operatorname{Re}\int_{\mathbb{R}^N}(2^*-2)|z|^{2^*-4}\operatorname{Re}(z\overline{w}_{\mu,\xi})z\overline{v}\,dx.
\end{aligned} \tag{43}$$

We know that $w_{\mu,\xi} = -\varepsilon L_{e^{i\sigma}z_{\mu,\xi}} G_1'(e^{i\sigma}z_{\mu,\xi}) + o(\varepsilon)$, and hence

$$\langle f_0''(z)\phi_{\mu,\xi}, v\rangle_E = -\langle G_1'(z), v\rangle_E, \quad \forall v \in E \tag{44}$$

where $\phi_{\mu,\xi} = \lim_{\epsilon\to 0} \frac{w_{\mu,\xi}}{\epsilon}$. This implies that $\phi_{\mu,\xi}$ solves

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{R}^N} \nabla\phi_{\mu,\xi}\cdot\overline{\nabla v}dx - \operatorname{Re}\int_{\mathbb{R}^N} |z|^{2^*-2}\phi_{\mu,\xi}\overline{v}\,dx - \operatorname{Re}\int_{\mathbb{R}^N}(2^*-2)|z|^{2^*-4}\operatorname{Re}(z\overline{\phi}_{\mu,\xi})z\overline{v}\,dx \\ = 2\operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}\nabla z\cdot A(x)\overline{v}dx + \operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}\operatorname{div}A\,z\overline{v}\,dx.\end{aligned}$$

Multiplying by $\mu^{\frac{N}{2}-1}$ and using the expression of $z = e^{i\sigma}\mu^{-\frac{(N-2)}{2}}z_0(\frac{x-\xi}{\mu})$, we get

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{R}^N}\mu^{\frac{N}{2}-1}\nabla_x\phi_{\mu,\xi}(x)\overline{\nabla v}dx - \operatorname{Re}\int_{\mathbb{R}^N}\mu^{-2}\left|z_0\left(\frac{x-\xi}{\mu}\right)\right|^{2^*-2}\mu^{\frac{N}{2}-1}\phi_{\mu,\xi}(x)\overline{v}dx \\ -\operatorname{Re}\int_{\mathbb{R}^N}(2^*-2)\mu^{N-4}\left|z_0\left(\frac{x-\xi}{\mu}\right)\right|^{2^*-4}\operatorname{Re}\left(e^{i\sigma}\mu^{-\frac{N}{2}+1}z_0\left(\frac{x-\xi}{\mu}\right)\mu^{\frac{N}{2}-1}\overline{\phi_{\mu,\xi}(x)}\right)\times \\ \times e^{i\sigma}\mu^{-\frac{N}{2}+1}z_0\left(\frac{x-\xi}{\mu}\right)\overline{v}dx \\ = 2\operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}e^{i\sigma}\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\cdot A(x)\overline{v}dx + \operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}\operatorname{div}A\,e^{i\sigma}z_0\left(\frac{x-\xi}{\mu}\right)\overline{v}dx.\end{aligned}$$

Using the expression of $\phi^*(\frac{x-\xi}{\mu}) = \mu^{\frac{N}{2}-1}\phi_{\mu,\xi}(x)$, we have

$$\begin{aligned}\operatorname{Re}\int\nabla_x\phi^*\left(\frac{x-\xi}{\mu}\right)\overline{\nabla v}dx - \operatorname{Re}\int\mu^{-2}\left|z_0\left(\frac{x-\xi}{\mu}\right)\right|^{2^*-2}\phi^*\left(\frac{x-\xi}{\mu}\right)\overline{v}dx \\ -\operatorname{Re}\int(2^*-2)\mu^{N-4}\left|z_0\left(\frac{x-\xi}{\mu}\right)\right|^{2^*-4}\operatorname{Re}\left(e^{i\sigma}\mu^{-\frac{N}{2}+1}z_0\left(\frac{x-\xi}{\mu}\right)\overline{\phi^*\left(\frac{x-\xi}{\mu}\right)}\right)\times \\ \times e^{i\sigma}\mu^{-\frac{N}{2}+1}z_0\left(\frac{x-\xi}{\mu}\right)\overline{v}dx \\ = 2\operatorname{Re}\int\frac{1}{i}e^{i\sigma}\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\cdot A(x)\overline{v}dx + \operatorname{Re}\int\frac{1}{i}\operatorname{div}A(x)\,e^{-i\sigma}z_0\left(\frac{x-\xi}{\mu}\right)\overline{v}dx.\end{aligned}$$

then, the change of variable $x = \mu y + \xi$ yields

$$\begin{aligned}\operatorname{Re}\int\mu^{-2}\nabla_y\phi^*(y)\overline{\nabla_y v(\mu y+\xi)}\mu^N dy - \operatorname{Re}\int\mu^{N-2}\,|\,z_0(y)\,|^{2^*-2}\,\phi^*(y)\overline{v(\mu y+\xi)}dy \\ -\operatorname{Re}\int(2^*-2)\mu^{N-4}\,|\,z_0(y)|^{2^*-4}\operatorname{Re}\left(e^{i\sigma}\mu^{2(-\frac{N}{2}+1)}z_0(y)\overline{\phi^*(y)}\right)e^{i\sigma}z_0(y)\overline{v(\mu y+\xi)}\mu^N dy \\ = 2\operatorname{Re}\int\frac{1}{i}e^{i\sigma}\nabla_y z_0(y)\cdot A(\mu y+\xi))\overline{v(\mu y+\xi)}\mu^{N-1}dy \\ + \operatorname{Re}\int\frac{1}{i}\operatorname{div}A(\mu y+\xi)\,e^{-i\sigma}z_0(y)\overline{v(\mu y+\xi)}\mu^N dy.\end{aligned}$$

Replacing $x = y$ and dividing by μ^{N-2}, it results

$$\begin{aligned} \operatorname{Re}\int_{\mathbb{R}^N} \nabla_x\phi^*(x)\overline{\nabla_x v(\mu x+\xi)}\,dx - \operatorname{Re}\int_{\mathbb{R}^N} |\, z_0(x)\,|^{2^*-2}\,\phi^*(x)\overline{v(\mu x+\xi)}\,dx \\ - \operatorname{Re}\int_{\mathbb{R}^N}(2^*-2)\,|\,z_0(x)|^{2^*-4}\operatorname{Re}\left(e^{i\sigma}z_0(x)\phi^*(x)\right)e^{i\sigma}z_0(x)\overline{v(\mu x+\xi)}\,dx \\ = 2\mu\operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}e^{i\sigma}\nabla_x z_0(x)\cdot A(\mu x+\xi))\overline{v(\mu x+\xi)}\,dx \\ + \mu^2\operatorname{Re}\int_{\mathbb{R}^N}\frac{1}{i}\operatorname{div}_y A(\mu x+\xi)\,e^{i\sigma}z_0(x)\overline{v(\mu x+\xi)}\,dx. \end{aligned}$$

This means that, if we write $\tau_{\mu,\xi}(x) = \mu x + \xi$,

$$\left\langle f_0''(e^{i\sigma}z_0)\phi^*, v\circ\tau_{\mu,\xi}\right\rangle = \int k_{\mu,\xi}\overline{v\circ\tau_{\mu,\xi}}$$

for all test function v, in particular that

$$f_0''(e^{i\sigma}z_0)\phi^* = k_{\mu,\xi}$$

where

$$k_{\mu,\xi}(x) = \frac{2}{i}\mu e^{i\sigma}\nabla_x z_0(x)\cdot A(\mu x+\xi) + \frac{1}{i}\mu^2 e^{i\sigma}\operatorname{div}_y A(\mu x+\xi)\,z_0(x).$$

We conclude that ϕ^* is a solution of

$$\phi^*(x) = L_{e^{i\sigma}z_0}k_{\mu,\xi}(x) \tag{45}$$

Our assumptions on A (i.e. **(A1)** and **(A2)**) imply immediately that

$$k_{\mu,\xi}\to 0 \quad \text{in } E \text{ as } \mu\to 0. \tag{46}$$

From the continuity of $L_{e^{i\sigma}z_0}$ we deduce that

$$\lim_{\mu\to 0+}\phi^* = \lim_{\mu\to 0+}L_{e^{i\sigma}z_0}k_{\mu,\xi} = 0. \tag{47}$$

This completes the proof of the Lemma. □

Lemma 4.6. *Under assumption **(A1)**, there holds*

$$\lim_{\mu+|\xi|\to+\infty} H_2(\mu,\xi) = 0,$$

where H_2 is defined in (36).

Proof. Firstly, assume that $\mu\to\overline{\mu}\in(0,+\infty)$ and $\mu+|\xi|\to+\infty$. We notice that

$$\begin{aligned} H_2(\mu,\xi) &= \frac{\mu^{-(N-2)}}{2}\int_{\mathbb{R}^N}|A(x)|^2 z_0^2\left(\frac{x-\xi}{\mu}\right)dx \\ &= \frac{\mu^{-(N-2)}}{2}\int_{|x|\le\frac{|\xi|}{2}}|A(x)|^2 z_0^2\left(\frac{x-\xi}{\mu}\right)dx \\ &\quad + \frac{\mu^{-(N-2)}}{2}\int_{|x|>\frac{|\xi|}{2}}|A(x)|^2 z_0^2\left(\frac{x-\xi}{\mu}\right)dx. \end{aligned}$$

Moreover,

$$
\begin{aligned}
&\frac{\mu^{-(N-2)}}{2}\int_{|x|\le\frac{|\xi|}{2}}|A(x)|^2 z_0^2\left(\frac{x-\xi}{\mu}\right)dx\\
&\quad\le \frac{\mu^{-(N-2)}}{2}||A||_\infty^2\omega_N\frac{|\xi|^N}{2^N}\sup_{|x|\le\frac{|\xi|}{2}} z_0^2\left(\frac{x-\xi}{\mu}\right)\\
&\quad= \frac{\mu^{-(N-2)}}{2}||A||_\infty^2\omega_N\frac{|\xi|^N}{2^N}\sup_{|x|\le\frac{|\xi|}{2}}\frac{k_N^2\mu^{2(N-2)}}{[\mu^2+|x-\xi|^2]^{N-2}}\\
&\quad\le \frac{\mu^{-(N-2)}}{2}||A||_\infty^2\omega_N\frac{|\xi|^N}{2^N}\sup_{|x|\le\frac{|\xi|}{2}}\frac{k_N^2}{\left[\mu^2+|\,|x|-|\xi|\,|^2\right]^{N-2}}\\
&\quad\le \frac{\mu^{-(N-2)}}{2}||A||_\infty^2\omega_N\frac{|\xi|^N}{2^N}\frac{k_N^2}{\left[\mu^2+\frac{|\xi|^2}{4}\right]^{N-2}},
\end{aligned}
$$

where ω_N is the measure of $S^{N-1}=\left\{x\in\mathbb{R}^N : |x|=1\right\}$. Since $N>4$, we infer

$$
\frac{k_N^2|\xi|^N}{\left[\mu^2+\frac{|\xi|^2}{4}\right]^{N-2}}\to 0 \quad \text{as } |\xi|\to+\infty.
$$

Finally, we deduce

$$
\frac{\mu^{-(N-2)}}{2}\int_{|x|\le\frac{|\xi|}{2}}|A(x)|^2 z_0^2\left(\frac{x-\xi}{\mu}\right)dx\to 0
$$

as $\mu\to\overline{\mu}$ and $|\xi|\to+\infty$.

On the other hand, we have

$$
\begin{aligned}
&\frac{\mu^{-(N-2)}}{2}\int_{|x|>\frac{|\xi|}{2}}|A(x)|^2 z_0^2\left(\frac{x-\xi}{\mu}\right)dx\\
&\quad\le \frac{\mu^{-(N-2)}}{2}||A||_\infty^2\int_{|x|>\frac{|\xi|}{2}} z_0^2\left(\frac{x-\xi}{\mu}\right)dx\\
&\quad= \frac{\mu^{N-(N-2)}}{2}||A||_\infty^2\int_{|\mu x+\xi|>\frac{|\xi|}{2}} z_0^2(x)dx.
\end{aligned}
$$

Since $z_0^2\in L^1(\mathbb{R}^N)$, we deduce that

$$
\frac{\mu^2}{2}||A||_\infty^2\int_{|\mu x+\xi|>\frac{|\xi|}{2}} z_0^2(x)dx\to 0
$$

as $\mu\to\overline{\mu}$ and $|\xi|\to+\infty$, and thus

$$
\frac{\mu^{-(N-2)}}{2}\int_{|x|>\frac{|\xi|}{2}}|A(x)|^2 z_0^2\left(\frac{x-\xi}{\mu}\right)dx\to 0
$$

as $\mu \to \overline{\mu}$ and $|\xi| \to +\infty$.
Finally, we can conclude that $H_2(\mu, \xi) \to 0$ as $\mu \to \overline{\mu}$ and $|\xi| \to +\infty$.
Conversely, assume that $\mu \to +\infty$. After a suitable change of variable, it results

$$H_2(\mu, \xi) = \frac{\mu^2}{2} \int_{\mathbb{R}^N} |A(\mu y + \xi)|^2 |z_0(y)|^2 dy.$$

By assumption **(A1)**, we can fix $1 < r < \frac{N}{2}$ such that $A^2 \in L^r(\mathbb{R}^N)$. Moreover, let be $s = \frac{r}{r-1}$. It is immediate to check that $2s > 2^*$ and then $z_0^{2s} \in L^1(\mathbb{R}^N)$. By (A1) and Holder inequality, we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} |A(\mu y + \xi)|^2 |z_0(y)|^2 dy \\ &\qquad \le \left(\int_{\mathbb{R}^N} |A(\mu y + \xi)|^{2r} dy \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^N} |z_0(y)|^{2s} dy \right)^{\frac{1}{s}} \\ &\qquad\qquad \le \mu^{-\frac{N}{r}} \left(\int_{\mathbb{R}^N} |A(y)|^{2r} dy \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^N} |z_0(y)|^{2s} dy \right)^{\frac{1}{s}}. \end{aligned}$$

As a consequence, by the above inequality, we infer for μ small

$$\begin{aligned} G_2(\mu, \xi) = \frac{\mu^2}{2} &\int_{\mathbb{R}^N} |A(\mu y + \xi)|^2 |z_0(y)|^2 dy \\ &\qquad \le \mu^{2-\frac{N}{r}} \left(\int_{\mathbb{R}^N} |A(y)|^{2r} dy \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^N} |z_0(y)|^{2s} dy \right)^{\frac{1}{s}}. \end{aligned}$$

Now, we notice that $r < \frac{N}{2}$ implies $2 - \frac{N}{r} < 0$ and thus by the above inequality we can conclude that $G_2(\mu, \xi)$ tends to 0 as $\mu \to +\infty$. $\square$

Arguing as before we can deduce the following result.

Lemma 4.7. *Under assumption* ***(V)****, there holds*

$$\lim_{\mu + |\xi| \to +\infty} \int_{\mathbb{R}^N} V(x) |z_{\mu,\xi}(x)|^2 \, dx = 0.$$

In order to describe the behavior of the term $\langle G_1'(z), \phi \rangle_E$ as $\mu + |\xi| \to +\infty$, we need the following lemma.

Lemma 4.8. *There is a constant $C_N > 0$ such that*

$$\| \phi \|_E \le C_N \quad \text{for all } \mu > 0 \text{ and for all } \xi \in \mathbb{R}^N. \tag{48}$$

Proof. We know that for all $\varepsilon > 0$ and all $z \in Z$

$$w(\varepsilon, z) = -L_z G_1'(z) + o(\varepsilon)$$

so that

$$\phi = \lim_{\varepsilon \to 0} \frac{w(\varepsilon, z)}{\varepsilon} = -L_z G_1'(z)$$

and

$$\|\phi\|_E \leq \|L_z\| \, \|G_1'(z)\| \, .$$

We claim that $\|L_z\|$ is bounded above by a constant independent of μ and ξ. Indeed:

$$\begin{aligned}
\|L_z\| &= \sup_{\|\varphi\|=1} \|L_z\varphi\| = \sup_{\substack{\|\varphi\|=1\\ \|\psi\|=1}} |\langle L_z\varphi, \psi\rangle| \\
&= \sup_{\substack{\|\varphi\|=1\\ \|\psi\|=1}} \Big| \int_{\mathbb{R}^N} \nabla\varphi \cdot \overline{\nabla\psi} - \mathrm{Re} \int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2} \varphi\overline{\psi} \\
&\qquad\qquad - (2^*-2) \int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-4} \mathrm{Re}(\overline{\varphi} z_{\mu,\xi}) \, \mathrm{Re}(\overline{\psi} z_{\mu,\xi}) \Big| \\
&\leq \sup_{\substack{\|\varphi\|=1\\ \|\psi\|=1}} \left(\int_{\mathbb{R}^N} |\nabla\varphi| \, |\overline{\nabla\psi}| + \mathrm{Re} \int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2} |\varphi| \, |\overline{\psi}| + (2^*-2) \int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2} |\varphi| \, |\overline{\psi}| \right) \\
&\leq \sup_{\substack{\|\varphi\|=1\\ \|\psi\|=1}} \left(\int_{\mathbb{R}^N} |\nabla\varphi| \, |\overline{\nabla\psi}| + (2^*-1) \int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*-2} |\varphi| \, |\overline{\psi}| \right) \\
&\leq \sup_{\substack{\|\varphi\|=1\\ \|\psi\|=1}} \left(\int_{\mathbb{R}^N} |\nabla\varphi|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla\psi|^2 \right)^{1/2} + (2^*-1) \left(\int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*} \right)^{(2^*-2)/2^*} \times \\
&\qquad\qquad \times \left(\int_{\mathbb{R}^N} |\varphi|^{2^*} \right)^{1/2^*} \left(\int_{\mathbb{R}^N} |\psi|^{2^*} \right)^{1/2^*}
\end{aligned}$$

We observe that

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |z_{\mu,\xi}|^{2^*} \right)^{1/2^*} &= \mu^{-\frac{(N-2)}{2}} \left(\int_{\mathbb{R}^N} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \right)^{1/2^*} \\
&= \mu^{-\frac{(N-2)}{2}} \left(\int_{\mathbb{R}^N} |z_0(y)|^{2^*} \mu^N \right)^{1/2^*} = \|z_0\|_{L^{2^*}} .
\end{aligned}$$

Hence

$$\begin{aligned}
\|L_z\| &\leq \sup_{\substack{\|\varphi\|=1\\ \|\psi\|=1}} \left(1 + (2^*-1) \| z_0\|_{L^{2^*}}^{(2^*-2)} \|\varphi\|_{L^{2^*}} \|\psi\|_{L^{2^*}} \right) \\
&\leq \sup_{\substack{\|\varphi\|=1\\ \|\psi\|=1}} \left(1 + (2^*-1) C_N' \|z_0\|_E^{(2^*-2)} \|\varphi\|_E \|\psi\|_E \right) \\
&\leq 1 + (2^*-1) C_N' \|z_0\|_E^{(2^*-2)} \equiv C_N^1
\end{aligned}$$

where C_N^1 is a constant independent from μ and ξ. At this point it results:

$$\|\phi\| \leq C_N^1 \, \|G_1'(z)\|$$

and we have to evaluate $\|G_1'(z)\|$:

$$\begin{aligned}
\|G_1'(z)\| &= \sup_{\|\varphi\|=1} |\langle G_1'(z),\varphi\rangle| \\
&= \sup_{\|\varphi\|=1} \left|\left(-\operatorname{Re}\int_{\mathbb{R}^N} \frac{\nabla z}{i}\cdot A(x)\overline{\varphi}\,dx - \operatorname{Re}\int_{\mathbb{R}^N} \frac{\nabla \phi}{i}\cdot A(x)\overline{z}\,dx\right)\right| \\
&\leq \sup_{\|\varphi\|=1} \left(\int_{\mathbb{R}^N} |\nabla z_{\mu,\xi}|\,|A(x)|\,|\overline{\varphi}|\;dx + \int_{\mathbb{R}^N} |\nabla\varphi|\,|A(x)|\,|z_{\mu,\xi}|\;dx\right) \\
&\leq \|A\|_{L^N} \sup_{\|\varphi\|=1} \left(\|z_0\|_E\,\|\varphi\|_E C_N''\right) \\
&\leq \|A\|_{L^N}\|z_0\|_E C_N'' \equiv C_N^2
\end{aligned}$$

with C_N^2 independent from μ and ξ.
Finally,

$$\|\,\phi\,\| \;\leq\; C_N^1 C_N^2 \equiv C_N$$

with C_N independent from μ and ξ and the lemma is proved. □

Remark 4.9. It is easy to check that $\|\phi^*\| = \|\phi\|$.

Lemma 4.10. *There holds*

$$\lim_{\mu+|\xi|\to+\infty} \langle G_1'(z),\phi\rangle_E = 0.$$

Proof. Firstly, assume that $\mu \to \overline{\mu} \in (0,+\infty)$ and $\mu+|\xi| \to +\infty$. We can write

$$\langle G_1'(z),\phi\rangle_E = \alpha_1+\alpha_2$$

where

$$\alpha_1 = -\operatorname{Re}\int_{\mathbb{R}^N} \frac{\nabla z}{i}\cdot A(x)\overline{\phi}\,dx \tag{49}$$

$$\alpha_2 = -\operatorname{Re}\int_{\mathbb{R}^N} \frac{\nabla \phi}{i}\cdot A(x)\overline{z}\,dx. \tag{50}$$

Using the expression of $z = e^{i\sigma}\mu^{-\frac{(N-2)}{2}} z_0(\frac{x-\xi}{\mu})$ and by assumption **(A1)** and the Hölder inequality we have:

$$\begin{aligned}
\alpha_1 &= -\operatorname{Re}\int_{\mathbb{R}^N} \frac{1}{i}\nabla_x e^{i\sigma}\mu^{-\frac{(N-2)}{2}} z_0\left(\frac{x-\xi}{\mu}\right)\cdot A(x)\overline{\phi}(x)\,dx \\
&\leq \mu^{-\frac{(N-2)}{2}}\left(\int_{\mathbb{R}^N}\left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 dx\right)^{1/2}\left(\int_{\mathbb{R}^N}\left(|A(x)|\,|\overline{\phi}|\right)^2 dx\right)^{1/2} \\
&\leq \mu^{-\frac{(N-2)}{2}}\,\|A\|_{L^N(\mathbb{R}^N)}\,\|\phi\|_{L^{2^*}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}\left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 dx\right)^{1/2}
\end{aligned}$$

We notice that

$$\int_{\mathbb{R}^N} \left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 dx = \int_{|x|\,\le\,|\xi|/2} \left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 dx + \int_{|x|\,>\,|\xi|/2} \left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 dx$$

and

$$\left|\nabla_x z_0\left(\tfrac{x-\xi}{\mu}\right)\right|^2 = \mu^{2(2-N)}(2-N)^2\kappa_N^2 \frac{|x-\xi|^2}{\left(\mu^2+|x-\xi|^2\right)^N}.$$

Moreover, setting $C_N^2 := (2-N)^2\kappa_N^2$,

$$\begin{aligned}
\int_{|x|\,\le\,|\xi|/2} \left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 dx &\le \omega_N \frac{|\xi|^N}{2^N} \sup_{|x|\,\le\,|\xi|/2} \left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 \\
&= \omega_N \frac{|\xi|^N}{2^N} \sup_{|x|\,\le\,|\xi|/2} \mu^{2(2-N)}(2-N)^2\kappa_N^2 \frac{|x-\xi|^2}{\left(\mu^2+|x-\xi|^2\right)^N} \\
&= \mu^{2(2-N)}\omega_N \frac{|\xi|^N}{2^N} \sup_{|x|\,\le\,|\xi|/2} \frac{C_N^2\,|x-\xi|^2}{\left(\mu^2+|x-\xi|^2\right)^N} \\
&\le \mu^{2(2-N)}\omega_N \frac{|\xi|^N}{2^N} \sup_{|x|\,\le\,|\xi|/2} \frac{C_N^2\,\big(\,|x|+|\xi|\,\big)^2}{\Big(\mu^2+\big|\,|x|-|\xi|\,\big|^2\Big)^N} \\
&\le \frac{9}{4}\omega_N \frac{|\xi|^N}{2^N} \frac{C_N^2\,|\xi|^2}{\left(\mu^2+|\xi|^2/4\right)^N}
\end{aligned}$$

where ω_N is the measure of $S^{N-1} = \big\{x \in \mathbb{R}^N : |x| = 1\big\}$. From $N > 4$, we infer

$$\frac{C_N^2\,|\xi|^{N+2}}{\left(\mu^2+|\xi|^2/4\right)^N} \to 0 \quad \text{as } |\xi| \to +\infty.$$

Finally, we deduce

$$\int_{|x|\,\le\,|\xi|/2} \left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 dx \to 0 \quad \text{as } \mu \to \bar{\mu} \in (0,+\infty),\ |\xi| \to +\infty.$$

On the other hand, we have

$$\int_{|x|\,>\,|\xi|/2} \left|\nabla_x z_0\left(\frac{x-\xi}{\mu}\right)\right|^2 dx \le \mu^{N-2} \int_{|\mu x+\xi|\,>\,|\xi|/2} \left|\nabla_x z_0(x)\right|^2 dx.$$

Since $|\nabla_x z_0|^2 \in L^1(\mathbb{R}^N)$, we deduce that

$$\mu^{N-2} \int_{|\mu x+\xi|\,>\,|\xi|/2} \left|\nabla_x z_0(x)\right|^2 dx \to 0$$

as $\mu \to \overline{\mu} \in (0, +\infty)$ and $|\xi| \to +\infty$ and thus

$$\int_{|x|\,>\,|\xi|/2} \left| \nabla_x z_0 \left(\frac{x-\xi}{\mu} \right) \right|^2 \, dx \,\to\, 0$$

and

$$\alpha_1 \leq \mu^{-\frac{(N-2)}{2}} \, \|A\|_{L^N(\mathbb{R}^N)} \, \|\phi\|_{L^{2^*}(\mathbb{R}^N)} \, \left(\int_{\mathbb{R}^N} \left| \nabla_x z_0 \left(\frac{x-\xi}{\mu} \right) \right|^2 \, dx \right)^{1/2} \,\to\, 0$$

as $\mu \to \overline{\mu} \in (0, +\infty)$ and $|\xi| \to +\infty$.

As regards α_2 we know that

$$\begin{aligned}
\alpha_2 &= -\operatorname{Re} \int_{\mathbb{R}^N} \frac{1}{i} \nabla_x \phi(x) \cdot A(x) e^{-i\sigma} \mu^{-\frac{(N-2)}{2}} z_0 \left(\frac{x-\xi}{\mu} \right) \, dx \\
&\leq \mu^{-\frac{(N-2)}{2}} \left(\int_{\mathbb{R}^N} |\, \nabla_x \phi(x) \cdot A(x)|^\beta \, dx \right)^{1/\beta} \left(\int_{\mathbb{R}^N} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx \right)^{1/2^*} \\
&\leq \mu^{-\frac{(N-2)}{2}} \, \|\, \phi \,\|_E \, \|\, A \,\|_{L^N} \left(\int_{\mathbb{R}^N} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx \right)^{1/2^*}
\end{aligned}$$

with $\beta = 2N/(N+2)$. We notice that

$$\begin{aligned}
\int_{\mathbb{R}^N} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx = \int_{|x|\,\leq\,|\xi|/2} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx \\
+ \int_{|x|\,>|\xi|/2} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{|x|\,\leq\,|\xi|/2} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx &\leq \omega_N \frac{|\xi|^N}{2^N} \sup_{|x|\,\leq\,|\xi|/2} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \\
&= \omega_N \frac{|\xi|^N}{2^N} \sup_{|x|\,\leq\,|\xi|/2} \mu^{2N} \kappa_N^{2^*} \frac{\mu^{2N} \kappa_N^{2^*}}{(\mu^2 + |x-\xi|^2)^N} \\
&\leq \mu^{2N} \omega_N \frac{|\xi|^N}{2^N} \sup_{|x|\,\leq\,|\xi|/2} \frac{\kappa_N^{2^*}}{\left(\mu^2 + |\,|x| - |\xi|\,|^2\right)^N} \\
&\leq \mu^{2N} \omega_N \frac{|\xi|^N}{2^N} \frac{\kappa_N^{2^*}}{(\mu^2 + |\xi|^2/4)^N}
\end{aligned}$$

where ω_N is the measure of $S^{N-1} = \left\{ x \,\in\, \mathbb{R}^N \,:\, |x| = 1 \right\}$. From $N > 4$, we infer

$$\frac{\kappa_N^{2^*} \, |\xi|^N}{(\mu^2 + |\xi|^2/4)^N} \,\to\, 0 \quad \text{as } |\xi| \to +\infty.$$

Finally, we deduce

$$\int_{|x|\,\leq\,|\xi|/2} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx \, \to \, 0$$

as $\mu \to \overline{\mu} \in (0,+\infty)$ and $|\xi| \to +\infty$.

On the other hand, we have

$$\int_{|x|\,>\,|\xi|/2} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx \, \leq \, \mu^N \int_{|\mu x+\xi|\,>\,|\xi|/2} |\, z_0(x)\,|^{2^*} \, dx.$$

Since $|\, z_0|^{2^*} \, \in \, L^1(\mathbb{R}^N)$, we deduce that

$$\mu^N \int_{|\mu x+\xi|\,>\,|\xi|/2} |\, z_0(x)\,|^{2^*} \, dx \, \to \, 0$$

as $\mu \to \overline{\mu} \in (0,+\infty)$ and $|\xi| \to +\infty$ and thus

$$\int_{|x|\,>\,|\xi|/2} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx \, \to \, 0$$

and

$$\alpha_2 \, \leq \, \mu^{-\frac{(N-2)}{2}} \, \|A\|_{L^N(\mathbb{R}^N)} \, \|\phi\|_E \left(\int_{\mathbb{R}^N} \left| z_0 \left(\frac{x-\xi}{\mu} \right) \right|^{2^*} \, dx \right)^{1/2^*} \quad \to \, 0$$

as $\mu \to \overline{\mu} \in (0,+\infty)$ and $|\xi| \to +\infty$.

Conversely, assume that $\mu \, \to \, +\infty$. Now it is convenient to write

$$\langle G_1'(z), \phi \rangle_E = \alpha_1 + \alpha_2$$

where

$$\alpha_1 = -\mu \operatorname{Re} \int_{\mathbb{R}^N} \frac{e^{i\sigma}}{i} \nabla_y z_0(y) \cdot A(\mu y + \xi) \overline{\phi^*(y)} \, dy$$

and

$$\alpha_2 = -\mu \operatorname{Re} \int_{\mathbb{R}^N} \frac{1}{i} \nabla_y \phi^*(y) \cdot A(\mu y + \xi) e^{-i\sigma} z_0(y) \, dy.$$

The Hölder inequality implies that

$$\alpha_1 \, \leq \, \mu \| \, \phi^* \, \|_{L^{2^*}} \left(\int_{\mathbb{R}^N} (\, \nabla_y z_0(y) \cdot A(\mu y + \xi))^\beta \, dy \right)^{1/\beta}$$

where $1/2^* + 1/\beta = 1$ so $\beta = 2N/(N+2)$. By assumptions **(A1)**, we can fix $r \in (1, (N+2)/2)$ such that $A^\beta \, \in \, L^r(\mathbb{R}^N)$. Moreover, let $s = r/(r-1)$. It is immediate to check that $\beta s \, > \, 2$

and then $|\nabla_y z_0|^{\beta s} \in L^1(\mathbb{R}^N)$. By **(A1)** and the Hölder inequality, we deduce that:

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^N} (\nabla_y z_0(y) \cdot A(\mu y + \xi))^\beta \, dy \right)^{1/\beta} \\
&\qquad \le \left(\int_{\mathbb{R}^N} (\nabla_y z_0(y))^{\beta s} \, dy \right)^{1/\beta s} \left(\int_{\mathbb{R}^N} (A(\mu y + \xi))^{\beta r} \, dy \right)^{1/\beta r} \\
&\qquad\qquad \le \mu^{-\frac{N}{\beta r}} \|\nabla_y z_0(y)\|_{L^{\beta s}} \left(\int_{\mathbb{R}^N} (A(\mu y + \xi))^{\beta r} \, dy \right)^{1/\beta r}
\end{aligned}
$$

As a consequence, by the above inequality, we infer for μ small:

$$
\begin{aligned}
\alpha_1 &\le \mu^{1-\frac{N}{\beta r}} \|\nabla_y z_0(y)\|_{L^{\beta s}} \left(\int_{\mathbb{R}^N} (A(\mu y + \xi))^{\beta r} \, dy \right)^{1/\beta r} \| \phi^* \|_{L^{2^*}} \\
&\le \mu^{1-\frac{N}{\beta r}} C'_N \|z_0\|_E \|A\|_{L^{\beta r}} \|\phi^*\|_E
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
\alpha_2 &\le \mu \left(\int_{\mathbb{R}^N} (|\nabla_y \phi^*(y)| \, |A(\mu y + \xi)| \, dy)^\beta \right)^{1/\beta} \left(\int_{\mathbb{R}^N} |z_0(y)|^{2^*} \, dy \right)^{1/2^*} \\
&\le \mu^{1-\frac{N}{\beta r}} \| z_0 \|_{L^{2^*}} \left(\int_{\mathbb{R}^N} (\nabla_y \phi^*(y))^{\beta s} \, dy \right)^{1/\beta s} \left(\int_{\mathbb{R}^N} (A(y))^{\beta r} \, dy \right)^{1/\beta r} \\
&\le \mu^{1-\frac{N}{\beta r}} C''_N (\| z_0 \|_E \, \| A \|_{L^{\beta r}} \, \| \phi^* \|_E)
\end{aligned}
$$

Since $\beta = 2N/(N+2)$, we deduce $1 - \frac{N}{\beta r} < 0$. The conclusion follows immediately from Lemma 4.8. ☐

Proposition 4.11. *Assume that there exists $\xi \in \mathbb{R}^N$ with $V(\xi) \neq 0$. Then*

$$
\lim_{\mu \to 0^+} \frac{\Gamma(\mu, \xi)}{\mu^2} = \frac{1}{2} V(\xi) \int |z_0|^2. \tag{51}
$$

In particular, Γ is a non-constant map.

Proof. If $V(\xi) \neq 0$ for some $\xi \in \mathbb{R}^N$, we can immediately check that $\Gamma(\mu, \xi)$ is not identically zero. More precisely, we prove that for every $\xi \in \mathbb{R}^N$ there holds

$$
\lim_{\mu \to 0^+} \frac{\Gamma(\mu, \xi)}{\mu^2} = \frac{1}{2} V(\xi) \int_{\mathbb{R}^N} |z_0|^2. \tag{52}
$$

Indeed, after a suitable change of variable,

$$
\begin{aligned}
\lim_{\mu \to 0^+} \frac{G_2(z_{\mu,\xi})}{\mu^2} &= \lim_{\mu \to 0^+} \frac{1}{2} \int_{\mathbb{R}^N} (|A(\mu y + \xi)|^2 |z_0(y)|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\mu y + \xi) |z_0(y)|^2 \, dy \\
&= \frac{1}{2} |A(\xi)|^2 \int_{\mathbb{R}^N} |z_0(y)|^2 \, dy + \frac{1}{2} V(\xi) \int_{\mathbb{R}^N} |z_0(y)|^2 \, dy. \qquad (53)
\end{aligned}
$$

To complete the proof of (52), we need to study $\lim_{\mu\to 0^+}\frac{1}{2\mu^2}\langle G_1'(z_{\mu,\xi}),\phi_{\mu,\xi}\rangle$.

In Lemma 4.5, we have showed that

$$\left\langle G_1'(e^{i\sigma}z_{\mu,\xi}),\phi_{\mu,\xi}\right\rangle = -\langle f_0''(z_{\mu,\xi}e^{i\sigma})\phi_{\mu,\xi},\phi_{\mu,\xi}\rangle = -\langle f_0''(z_0e^{i\sigma})\phi^*_{\mu,\xi},\phi^*_{\mu,\xi}\rangle$$

where $\phi^*_{\mu,\xi}(\frac{x-\xi}{\mu}) = \mu^{N/2-1}\phi_{\mu,\xi}(x)$ and $f_0''(z_0e^{i\sigma})\phi^*_{\mu,\xi} = k_{\mu,\xi}$, where

$$k_{\mu,\xi}(y) = \frac{2}{i}\mu e^{i\sigma}\nabla_y z_0(y)\cdot A(\mu y+\xi) + \frac{\mu^2}{i}e^{i\sigma}\operatorname{div}_y A(\mu y+\xi)z_0(y).$$

As $\mu\to 0^+$, we have $k_{\mu,\xi}\to k_\xi$, where

$$k_\xi(x) := \frac{2}{i}\ e^{i\sigma}\nabla_y z_0(y)\cdot A(\xi).$$

Let us define $\psi_\xi(x) = \lim_{\mu\to 0^+}\frac{L_{z_0}k_{\mu,\xi}}{\mu} = \lim_{\mu\to 0^+}\frac{\phi^*_{\mu,\xi}}{\mu}$. We have that

$$f_0''(z_0e^{i\sigma})\psi_\xi = \frac{2}{i}\ e^{i\sigma}\nabla_x z_0(y)\cdot A(\xi). \tag{54}$$

Setting $g_\xi(x) = e^{-i\sigma}\psi_\xi(x)$, we have that for any $\phi\in D^{1,2}(\mathbb{R}^N,\mathbb{R})$

$$\langle f_0''(z_0e^{i\sigma})e^{i\sigma}g_\xi, e^{i\sigma}\phi\rangle = \operatorname{Re}\int\frac{2}{i}\ e^{i\sigma}\nabla_y z_0(y)\cdot A(\xi)e^{-i\sigma}\phi\,dx = 0.$$

This means that for any $\phi\in\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R})$

$$\begin{aligned}
0 &= \langle f_0''(z_0e^{i\sigma})e^{i\sigma}g_\xi, e^{i\sigma}\phi\rangle\\
&= \operatorname{Re}\int\nabla(e^{i\sigma}g_\xi)\cdot\overline{\nabla(e^{i\sigma}\phi)} - \operatorname{Re}\int|z_0|^{2^*-2}e^{i\sigma}g_\xi\overline{e^{i\sigma}\phi}\\
&- \operatorname{Re}(2^*-2)\int|z_0|^{2^*-4}\operatorname{Re}(e^{i\sigma}z_0e^i\sigma g_\xi)e^{i\sigma}z_0\overline{e^{i\sigma}\phi}\\
&= \operatorname{Re}\int\nabla(g_\xi)\cdot\overline{\nabla\phi} - \operatorname{Re}\int|z_0|^{2^*-2}g_\xi\overline{\phi}\\
&- \operatorname{Re}(2^*-2)\int|z_0|^{2^*-4}\operatorname{Re}(z_0g_\xi)z_0\overline{\phi}\\
&= \int\nabla(\operatorname{Re}g_\xi)\cdot\overline{\nabla\phi} - \int|z_0|^{2^*-2}\operatorname{Re}g_\xi\overline{\phi}\\
&- (2^*-2)\int|z_0|^{2^*-4}\operatorname{Re}g_\xi {z_0}^2\overline{\phi}\\
&= \langle f_0''(z_0)\operatorname{Re}g_\xi,\phi\rangle.
\end{aligned}$$

It follows that $\operatorname{Re}g_\xi = 0$ as $\phi_{\mu,\xi}\in\left(T_{e^{i\sigma}z_{\mu,\xi}}Z\right)^\perp$. Therefore $\psi_\xi(x) = ie^{i\sigma}r_\xi(x)$ with $r_\xi\in D^{1,2}(\mathbb{R}^N,\mathbb{R})$. Now we test (54) against functions of the type $ie^{i\sigma}\omega(x)$, $\omega\in D^{1,2}(\mathbb{R}^N,\mathbb{R})$.

It results:

$$
\begin{aligned}
\mathrm{Re}\int \frac{2}{i}\, e^{i\sigma}\nabla_x z_0(x)\cdot A(\xi)\overline{ie^{i\sigma}w} &= \left\langle f_0''(z_0e^{i\sigma})\psi_\xi, ie^{i\sigma}w\right\rangle \\
&= \mathrm{Re}\int \nabla r_\xi\cdot\nabla w - \mathrm{Re}\int |z_0|^{2^*-2}r_\xi w \\
&- \mathrm{Re}(2^*-2)\int |z_0|^{2^*-4}\,\mathrm{Re}(iz_0r_\xi)z_0iw
\end{aligned}
$$

or equivalently

$$
\mathrm{Re}\int \nabla r_\xi\cdot\nabla w - \mathrm{Re}\int |z_0|^{2^*-2}r_\xi w = -\,\mathrm{Re}\int 2\nabla_x z_0(x)\cdot A(\xi)w.
$$

We deduce that r_ξ satisfies the equation

$$
-\Delta r_\xi(x) - |z_0|^{2^*-2}r_\xi(x) = -2\nabla z_0\cdot A(\xi). \tag{55}
$$

We notice that the function $u(x)=z_0(x)A(\xi)\cdot x$ solves the equation (55), as $\Delta u = \Delta z_0 A(\xi)\cdot x + z_0\Delta(A(\xi)\cdot x) + 2\nabla z_0\cdot\nabla(A(\xi)\cdot x) = \Delta z_0 A(\xi)\cdot x + 2\nabla z_0\cdot A(\xi)$.

Since $iz_0(x)(A(\xi)|x)e^{i\sigma}$ belongs to $(T_{e^{i\sigma}z_0}Z)^\perp$, we deduce that $\psi_\xi(x) = ie^{i\sigma}z_0(x)A(\xi)\cdot x$ and thus

$$
\begin{aligned}
\lim_{\mu\to 0^+}\frac{1}{2}\frac{\langle G_1'(z_{\mu,\xi}),\phi_{\mu,\xi}\rangle}{\mu^2} &= -\,\mathrm{Re}\int_{\mathbb{R}^N}\frac{1}{i}\, e^{i\sigma}\nabla_y z_0(y)\cdot A(\xi)\ \overline{ie^{i\sigma}z_0A(\xi)\cdot x}\,dx \\
&= \int_{\mathbb{R}^N}\ \nabla_y z_0(y)\cdot A(\xi)z_0\ A(\xi)\cdot x\,dx.
\end{aligned}
$$

Since we have

$$
\int_{\mathbb{R}^N}\nabla_y z_0(y)\cdot A(\xi)z_0A(\xi)\cdot x\,dx = -\int_{\mathbb{R}^N}\nabla_y z_0(y)\cdot A(\xi)z_0A(\xi)\cdot x\,dx - \int_{\mathbb{R}^N}|A(\xi)|^2z_0^2\,dx,
$$

we conclude that

$$
\lim_{\mu\to 0^+}\frac{1}{2}\frac{\langle G_1'(z_{\mu,\xi}),\phi_{\mu,\xi}\rangle}{\mu^2} = \int_{\mathbb{R}^N}\ \nabla_y z_0(y)\cdot A(\xi)z_0A(\xi)\cdot x\,dx = -\frac{1}{2}\int_{\mathbb{R}^N}|A(\xi)|^2z_0^2\,dx. \tag{56}
$$

Therefore we have that

$$
\lim_{\mu\to 0^+}\frac{\Gamma(\mu,\xi)}{\mu^2} = \lim_{\mu\to 0^+}\frac{1}{\mu^2}\left(G_2(\mu,\xi)+\frac{1}{2}\langle G_1'(z_{\mu,\xi}),\phi_{\mu,\xi}\rangle\right) = \frac{1}{2}V(\xi)\int_{\mathbb{R}^N}|z_0|^2.
$$

□

Remark 4.12. The presence of a non-trivial potential V is crucial in the previous Proposition. Otherwise, from (53) and (56) we would simply get that $\lim_{\mu\to 0^+}\frac{\Gamma(\mu,\xi)}{\mu^2}=0$, and Γ might still be a constant function. Hence V is in competition with A. It would be interesting to investigate the case in which $V=0$ identically. We conjecture that some additional assumptions on the shape of A should be made.

5 Proof of the main result

We recall the following abstract theorem from [4]. See also [5].

Theorem 5.1. *Assume that there exist a set $A \subseteq Z$ with compact closure and $z_0 \in A$ such that*

$$\Gamma(z_0) < \inf_{z \in \partial A} \Gamma(z) \; (\textit{resp.}\, \Gamma(z_0) > \sup_{z \in \partial A} \Gamma(z)).$$

Then, for ε small enough, f_ε has at least a critical point $u_\varepsilon \in Z_\varepsilon$ such that

$$f_0(z) + \varepsilon^2 \inf_A \Gamma + o(\varepsilon^2) \leq f_\varepsilon(u_\varepsilon) \leq f_0(z) + \varepsilon^2 \sup_{\partial A} \Gamma + o(\varepsilon^2)$$

$$(\textit{resp.}\, f_0(z) + \varepsilon^2 \inf_{\partial A} \Gamma + o(\varepsilon^2) \leq f_\varepsilon(u_\varepsilon) \leq f_0(z) + \varepsilon^2 \sup_A \Gamma + o(\varepsilon^2)).$$

Furthermore, up to a subsequence, there exists $\overline{z} \in A$ such that $u_{\varepsilon_n} \to \overline{z}$ in E as $\varepsilon_n \to 0$.

We can finally prove our main existence result for equation (3). According to Remark 1.1, we will use the term *solution* rather than the more precise S^1*–orbit of solutions.*

Theorem 5.2. *Retain assumptions **(N), (A1–2), (V)**. Assume that $V(\xi) \neq 0$ for some $\xi \in \mathbb{R}^N$. Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ equation* (3) *possesses at least one solution $u_\varepsilon \in E$. If V is a changing sign function, then there exists two solutions of equation* (3).

Proof. Under our assumptions, the Melnikov function Γ, extended across the hyperplane $\{\mu = 0\}$ by reflection, is not constant and possesses at least a critical point (either a minimum or a maximum point). We can therefore invoke Theorem 5.1 to conclude that there exists at least one solution u_ε to (3), provided ε is small enough. If there exist points $\xi_i \in \mathbb{R}^N$, $i = 1, 2$, such that $V(\xi_1)V(\xi_2) < 0$, then it follows from the previous Proposition that Γ must change sign near $\{\mu = 0\}$. In particular, it must have *both* a minimum *and* a maximum. Hence there exist two different solutions to (3). $\square$

Remark 5.3. Consider equation (3). It is clear that our main theorem still applies for any $\alpha \in [1, 2)$. Indeed, in the expansion (31), the lowest order term in ε is

$$\varepsilon^\alpha \int_{\mathbb{R}^N} V z^2 \, dx,$$

and consequently the magnetic potential A no longer affects the finite-dimensional reduction. In some sense, we have treated with the more all the details the "worst" situation in the range $1 \leq \alpha \leq 2$.

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